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$(\mathbb{Z}_2)^k$-actions and the minimal data of the normal bundle ☆

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ABSTRACT

Let $J^r_{n,k}$ denote the group of the n-dimensional unoriented cobordism classes containing a representative that admits a $(\mathbb{Z}_2)^k$-action with the fixed point set of constant dimension $n-r$ and $J^r_{*,k} = \sum_{n\geqslant r} J^r_{n,k}$. In this paper, we introduce the minimal data $\|J^r_{*,k}\|$ of the normal bundle which reflects the complexity of the $(\mathbb{Z}_2)^k$-action. Such minimal data or their upper bounds are determined for some r and k.

1. Introduction

Let MO_n denote the unoriented cobordism group and $MO_* = \sum_{n\geqslant 0} MO_n$ the unoriented cobordism ring [1], and let $J^r_{n,k}$ denote the group of the n-dimensional unoriented cobordism classes containing a representative that admits a $(\mathbb{Z}_2)^k$-action with the fixed point set of constant dimension $n-r$. Then $J^r_{*,k} = \sum_{n\geqslant r} J^r_{n,k}$ forms an ideal in MO_*.

We write $(\mathbb{Z}_2)^k$ for the group being generated by the elements $t_1, t_2, \dots, t_k$ subject to the relation $t_i^2 = 1$ and $t_i t_j = t_j t_i$ $(i, j = 1, 2, \dots, k)$, and $\mathrm{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)$ for the set of homomorphisms $(\mathbb{Z}_2)^k \to \mathbb{Z}_2 = \{1, -1\}$, consisting of 2^k distinct homomorphisms which we label f_i, $i = 0, 1, \dots, 2^k - 1$, where f_0 is the trivial homomorphism. Thus if $1 \leqslant i \leqslant 2^k - 1$, there is a generator t_j in $(\mathbb{Z}_2)^k$ such that $f_i(t_j) = -1$. We note that for $1 \leqslant i \leqslant 2^k - 1$, the kernel of the homomorphism $\ker f_i \cong (\mathbb{Z}_2)^{k-1}$.

Every irreducible representation of $(\mathbb{Z}_2)^k$ is one-dimensional and has the form $\lambda : (\mathbb{Z}_2)^k \times R \to R$ with $\lambda(t, y) = f_i(t)y$ for some f_i. Hence if ε is a real vector bundle with a $(\mathbb{Z}_2)^k$-action covering a compact Hausdorff space with a $(\mathbb{Z}_2)^k$-action, there are 2^k subbundles ε_i of ε with $\varepsilon = \varepsilon_0 \oplus \varepsilon_1 \oplus \cdots \oplus \varepsilon_{2^k-1}$ such that the action of t_j on ε corresponds to $(f_0(t_j)y_0, f_1(t_j)y_1, \dots, f_{2^k-1}(t_j)y_{2^k-1})$, where $y_i \in E(\varepsilon_i)$, the total space of ε_i, $i = 0, 1, \dots, 2^k - 1$.

Let $T : (\mathbb{Z}_2)^k \times M^n \to M^n$ be a smooth $(\mathbb{Z}_2)^k$-action with the fixed point set F of constant dimension $n-r$. Then F is a disjoint union of closed submanifolds of M^n. Let N_i represent the set of fixed points of $\ker f_i$ and $T_j = T(t_j, -)$. If $t_j \notin \ker f_i$, from [3] we know that F is the fixed point set of the involution $(N_i, T_j|_{N_i})$. The normal bundle ε to the fixed point set F of (M^n, T) can be decomposed as $\varepsilon = \varepsilon_1 \oplus \varepsilon_2 \oplus \cdots \oplus \varepsilon_{2^k-1}$, and $\varepsilon_i \to F$ is justly the normal bundle to the fixed point set F of the involution $(N_i, T_j|_{N_i})$. Since $f_0 = 1$, ε_0 is the trivial zero-dimensional bundle and may be ignored. In this way, F and the ordered set of the $2^k - 1$ vector bundles $\{\varepsilon_i\}_{i=1}^{2^k-1}$ constitute the fixed point data of the action on M^n.

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Let $I = \{(a_1, a_2, \dots, a_{2^k-1}) \mid \sum_{i=1}^{2^k-1} a_i = r,\ a_i$ is a non-negative integer$\}$. Then I is said to be the set of total $2^k - 1$ partitions of r.

For $A \subseteq I$, let $J^r_{n,k}(A)$ denote the set of the classes represented by a manifold M^n with a $(\mathbb{Z}_2)^k$-action having the fixed point data $(F^{n-r}, \{\varepsilon_i\}_{i=1}^{2^k-1})$ such that for each component F_j of F^{n-r},

$$\big(\dim(\varepsilon_1|_{F_j}), \dim(\varepsilon_2|_{F_j}), \dots, \dim(\varepsilon_{2^k-1}|_{F_j})\big) \in A.$$

Then $J^r_{*,k}(A) = \sum_{n \geqslant r} J^r_{n,k}(A)$ forms an ideal in MO_*.

Let $\tau_1 = (2,1,0)$, $\tau_2 = (1,0,2)$, $\tau_3 = (0,2,1)$, $\tau_4 = (1,2,0)$, $\tau_5 = (0,1,2)$ and $\tau_6 = (2,0,1)$. For $A = \{\tau_1,\tau_2,\tau_3\}$, $\{\tau_2,\tau_5,\tau_6\}$, $\{\tau_4,\tau_5,\tau_6\}$, $\{\tau_1,\tau_2,\tau_6\}$, $\{\tau_3,\tau_4,\tau_5\}$, $\{\tau_1,\tau_4,\tau_6\}$, $\{\tau_1,\tau_3,\tau_4\}$ and $\{\tau_2,\tau_3,\tau_5\}$, Pergher showed $J^3_{*,2}(A) = 0$ in [2]. For $A = \{(2,1,0),(2,0,1),(1,1,1)\}$, Wu and Guo computed the ideal $J^3_{*,2}(A)$ in [11]. For $A = \{(2,2,0),(2,0,2),(0,2,2)\}$, Wu and Guo determined the ideal $J^4_{*,2}(A)$ in [12].

In another way, one may consider to study $J^r_{*,k}(A)$. Let $\|A\|$ represent the number of the elements of A. The minimal data of the normal bundle for $J^r_{*,k}$ is defined by

$$\big\|J^r_{*,k}\big\| = \min\big\{\|A\| \bigm| J^r_{*,k}(A) = J^r_{*,k},\ A \subseteq I\big\},$$

which reflects the complexity of the $(\mathbb{Z}_2)^k$-actions. It is interesting to determine $\|J^r_{*,k}\|$ or a smaller upper bound of $\|J^r_{*,k}\|$.

The main results of this paper are

Theorem 1.1. $\|J^2_{*,k}\| = 3$, $k \geqslant 2$.

Theorem 1.2.

(1) $\|J^3_{*,k}\| = 3$, $k = 2$,
(2) $\|J^3_{*,k}\| \leqslant 3$, $k \geqslant 3$.

Theorem 1.3.

$$\big\|J^r_{*,k}\big\| \leqslant \begin{cases} 8, & r = 4,\ k \geqslant 2, \\ 9, & r = 5,\ k = 2. \end{cases}$$

2. Preliminary

If an n-dimensional cobordism class $[M^n]$ can be expressed as a sum of products of lower-dimensional cobordism classes in MO_*, then the class is called decomposable; otherwise it is indecomposable. Let $RP(n)$ be the real projective space of dimension n and $\chi : MO_* = \sum MO_n \to \mathbb{Z}_2$ the mod 2 Euler characteristic. Throughout this paper w denotes the total Stiefel–Whitney class and w_i the i-th Stiefel–Whitney class. $\equiv$ denotes congruence mod 2. Binomial coefficient are $\binom{m}{n} = m!/(n!(m-n)!)$.

Lemma 2.1. *([4]) Let $RP(n_1, n_2, \dots, n_l)$ be the projective space bundle of $\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l$ of $RP(n_1) \times RP(n_2) \times \cdots \times RP(n_l)$, where λ_i is the pullback of the canonical line bundle over the i-th factor. Then for $l > 1$, $[RP(n_1, n_2, \dots, n_l)]$ is indecomposable in MO_* if and only if*

$$s = \binom{m}{n_1} + \binom{m}{n_2} + \cdots + \binom{m}{n_l} \equiv 1 \pmod 2,$$

where $m = l + \sum_{i=1}^{l} n_i - 2$.

The manifold $RP(n_1, n_2, \dots, n_l)$ has dimension $m + 1$. If $n_{i+1} = n_{i+2} = \cdots = n_l = 0$, $RP(n_1, n_2, \dots, n_l)$ will sometimes be written as $RP(n_1, \dots, n_i; l)$.

Lemma 2.2. *([5]) If $m = \sum_{i=1}^{l} m_i 2^i$, and $n = \sum_{i=1}^{l} n_i 2^i$ with $0 \leqslant m_i, n_i \leqslant 1$, then*

$$\binom{m}{n} \equiv 1 \pmod 2 \iff \forall i,\ n_i \leqslant m_i.$$

Lemma 2.3. *([6]) If $0 < r < 2^k$, then*

$$J^r_{*,k} = \begin{cases} (0), & r = 1, \\ \sum_{n=r}^{\infty} MO_n, & r \text{ even}, \\ \sum_{n=r}^{\infty} MO_n \cap \ker\chi, & r > 1, \text{ odd}. \end{cases}$$

Let I, I' be the set of the total $2^k - 1$ partitions of r and r' respectively.

$$P = \{(a_{1,i}, a_{2,i}, \dots, a_{2^k-1,i}) \mid i = 1, \dots, m\} \subseteq I,$$
$$Q = \{(b_{1,j}, b_{2,j}, \dots, b_{2^k-1,j}) \mid j = 1, \dots, n\} \subseteq I'.$$

Let $\sigma = \begin{pmatrix} 1 & 2 & \cdots & 2^k-1 \\ n_1 & n_2 & \cdots & n_{2^k-1} \end{pmatrix} \in S_{2^k-1}$, where S_{2^k-1} is the symmetric group on $2^k - 1$ symbols, $\sigma P = \{(a_{n_1,i}, a_{n_2,i}, \dots, a_{n_{2^k-1},i}) \mid i = 1, \dots, m\}$, $P + Q = \{(a_{1,i} + b_{1,j}, a_{2,i} + b_{2,j}, \dots, a_{2^k-1,i} + b_{2^k-1,j}) \mid i = 1, 2, \dots, m;\ \ j = 1, 2, \dots, n\}$, and $P \oplus \theta = \{(a_{1,i}, a_{2,i}, \dots, a_{2^k-1,i}, \underbrace{0, 0, \dots, 0}_{2^{k+1}-2^k}) \mid i = 1, 2, \dots, m\}$. Then we have

Lemma 2.4.

(1) $J^r_{*,k}(P) = J^r_{*,k}(\sigma P)$, $\forall \sigma \in S_{2^k-1}$,
(2) $J^r_{*,k}(P) \cdot J^{r'}_{*,k}(Q) \subseteq J^{r+r'}_{*,k}(P + Q)$,
(3) $J^r_{*,k}(P) \subseteq J^r_{*,k+1}(P \oplus \theta)$.

Proof. (1) Let $\sigma = \begin{pmatrix} 1 & 2 & \cdots & 2^k-1 \\ n_1 & n_2 & \cdots & n_{2^k-1} \end{pmatrix}$. If M^n admits a $(\mathbb{Z}_2)^k$-action with the fixed point data $(F^{n-r}, \{\varepsilon_i\}_{i=1}^{2^k-1})$ such that for each component F_j of F^{n-r},

$$\left(\dim(\varepsilon_1|_{F_j}), \dim(\varepsilon_2|_{F_j}), \dots, \dim(\varepsilon_{2^k-1}|_{F_j})\right) \in P,$$

then

$$\left(\dim(\varepsilon_{n_1}|_{F_j}), \dim(\varepsilon_{n_2}|_{F_j}), \dots, \dim(\varepsilon_{n_{2^k-1}}|_{F_j})\right) \in \sigma P.$$

So $J^r_{*,k}(P) \subseteq J^r_{*,k}(\sigma P)$. Similarly $J^r_{*,k}(\sigma P) \subseteq J^r_{*,k}(\sigma^{-1}(\sigma P)) = J^r_{*,k}(P)$, therefore $J^r_{*,k}(P) = J^r_{*,k}(\sigma P)$.

(2) For $[M] \in J^r_{*,k}(P)$ and $[M'] \in J^{r'}_{*,k}(Q)$, by using the diagonal rule we can define a $(\mathbb{Z}_2)^k$-action on $M \times M'$ such that $[M \times M'] \in J^{r+r'}_{*,k}(P + Q)$.

(3) Let M^n admit a $(\mathbb{Z}_2)^k$-action with the fixed point data $(F^{n-r}, \{\varepsilon_i\}_{i=1}^{2^k-1})$ such that for each component F_j of F^{n-r},

$$\left(\dim(\varepsilon_1|_{F_j}), \dim(\varepsilon_2|_{F_j}), \dots, \dim(\varepsilon_{2^k-1}|_{F_j})\right) \in P.$$

By adding a new generator t_{k+1} acting as the identity on M, a $(\mathbb{Z}_2)^k$-action on M can be extended to a $(\mathbb{Z}_2)^{k+1}$-action with the same fixed point set. $\mathrm{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)$ consists of 2^k distinct homomorphisms which we label f_i, $i = 0, 1, \dots, 2^k - 1$. $\mathrm{Hom}((\mathbb{Z}_2)^{k+1}, \mathbb{Z}_2)$ consists of 2^{k+1} distinct homomorphisms which we label f'_j, $j = 0, 1, \dots, 2^{k+1} - 1$. Let $f'_j(t_{k+1}) = 1 (j = 0, 1, \dots, 2^k - 1)$, $f'_j(t_{k+1}) = -1 (j = 2^k, 2^k + 1, \dots, 2^{k+1} - 1)$. Then the group $\ker f_i = \ker f'_i$ for $i = 0, 1, \dots, 2^k - 1$.

Since t_{k+1} acts as the identity on M, if $f'_j(t_{k+1}) = -1$, then $N(\ker f'_j) = F$. If $f'_j(t_{k+1}) = 1$, then $N(\ker f'_j) = N(\ker f_j)$, where $N(\ker f'_j)$ represents the set of fixed points of $\ker f'_j$.

Let ε'_i be the normal bundle of F in $N(\ker f'_i)$. Then for each component F_l of F,

$$\begin{aligned} &\left(\dim \varepsilon'_1|_{F_l}, \dim \varepsilon'_2|_{F_l}, \dots, \dim \varepsilon'_{2^k-1}|_{F_l}, 0, 0, \dots, 0\right) \\ &\quad = (\dim \varepsilon_1|_{F_l}, \dim \varepsilon_2|_{F_l}, \dots, \dim \varepsilon_{2^k-1}|_{F_l}, 0, 0, \dots, 0) \in P \oplus \theta. \end{aligned}$$

So $J^r_{*,k}(P) \subseteq J^r_{*,k+1}(P \oplus \theta)$. □

Lemma 2.5. *If $0 < r < 2^k$, then $\|J^r_{*,k+1}\| \leqslant \|J^r_{*,k}\|$.*

Proof. Let $J^r_{*,k}(A) = J^r_{*,k}$ and $\|A\| = \|J^r_{*,k}\|$. By Lemma 2.4, $J^r_{*,k}(A) \subseteq J^r_{*,k+1}(A \oplus \theta)$. According to Lemma 2.3, $J^r_{*,k} = J^r_{*,k+1}$, so $J^r_{*,k+1}(A \oplus \theta) \subseteq J^r_{*,k+1} = J^r_{*,k} = J^r_{*,k}(A) \subseteq J^r_{*,k+1}(A \oplus \theta)$, $J^r_{*,k+1}(A \oplus \theta) = J^r_{*,k+1}$. Hence $\|J^r_{*,k+1}\| \leqslant \|A\| = \|J^r_{*,k}\|$. □

3. $\| J^2_{*,k} \|$ ($k \geqslant 2$)

Lemma 3.1. *$[RP(n_1, n_2, n_3)] \in J^2_{*,2}(A)$ for $A = \{(1,0,1), (1,1,0), (0,1,1)\}$.*

Proof. Let

$$T_1[y_1, y_2, y_3] = [-y_1, \quad y_2, \quad y_3],$$
$$T_2[y_1, y_2, y_3] = [-y_1, \quad -y_2, \quad y_3].$$

Then T_1, T_2 define a $(\mathbb{Z}_2)^2$-action on $RP(n_1, n_2, n_3)$ with the fixed point set

$$F = RP(\lambda_1) \cup RP(\lambda_2) \cup RP(\lambda_3)$$
$$= RP(n_1) \times RP(n_2) \times RP(n_3) \times \{e_1, e_2, e_3\},$$

and

$$F_{T_1} = RP(\lambda_1) \cup RP(\lambda_2 \oplus \lambda_3),$$
$$F_{T_2} = RP(\lambda_1 \oplus \lambda_2) \cup RP(\lambda_3),$$
$$F_{T_1 T_2} = RP(\lambda_1 \oplus \lambda_3) \cup RP(\lambda_2),$$

where F_{T_1}, F_{T_2} and $F_{T_1 T_2}$ denotes the fixed point set of T_1, T_2, and $T_1 T_2$ respectively. The components of F correspond to $A = \{(1,0,1), (1,1,0), (0,1,1)\}$. Hence $[RP(n_1, n_2, n_3)] \in J^2_{*,2}(A)$. $\square$

Proof of Theorem 1.1. From Theorem 8.1 in [4], there are indecomposable manifolds $RP(n_1, n_2, n_3)$ in every dimension. According to Theorem 3.2 in [2] and Lemma 3.1, $J^2_{*,2}(A) = J^2_{*,2} = \sum_{n \geqslant 2}^{\infty} MO_n$, so $\| J^2_{*,2} \| \leqslant 3$. From Lemma 2.5, we obtain $\| J^2_{*,k} \| \leqslant 3$, $k \geqslant 2$.

Next we show $\| J^2_{*,k} \| \geqslant 3$, $k \geqslant 2$.

By Lemma 2.3, we know $[RP(2)] \in J^2_{*,k}$. Let $T : (\mathbb{Z}_2)^k \times RP(2) \to RP(2)$ be a smooth $(\mathbb{Z}_2)^k$-action on $RP(2)$ with isolated fixed points. $(\mathbb{Z}_2)^k$ is generated by k commuting involutions $t_1, t_2, \ldots, t_k$. $\mathrm{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)$ consists of 2^k distinct homomorphisms which we label f_i, $i = 0, 1, \ldots, 2^k - 1$, we agree to let $f_0 = 1$. Let N_i be the fixed point set of $\ker f_i$. Every irreducible representation of $(\mathbb{Z}_2)^k$ is one-dimensional and has the form $\lambda : (\mathbb{Z}_2)^k \times R \to R$ with $\lambda(t, y) = f_i(t) y$ for some f_i. There are 2^k one-dimensional irreducible representations which we label Y_i: $t_j(s) = f_i(t_j)s$, $0 \leqslant i \leqslant 2^k - 1$, $1 \leqslant j \leqslant k$. Following [1], by $R_*((\mathbb{Z}_2)^k)$ we denote the unoriented representation algebra of the group $(\mathbb{Z}_2)^k$. Then $R_*((\mathbb{Z}_2)^k)$ is the polynomial algebra $\mathbb{Z}_2[Y_0, Y_1, \ldots, Y_{2^k-1}]$ (see [1]). If y is an isolated fixed point of the action $(RP(2), T)$, then the local representation class $X(y)$ at y can be decomposed by

$$X(y) = Y_0^{p_0} Y_1^{p_1} \ldots Y_{2^k-1}^{p_{2^k-1}}, \qquad p_0 + p_1 + \cdots + p_{2^k-1} = 2,$$

where p_i is the dimension of the component of N_i containing y. If $y \notin N_i$, then $p_i = 0$. By Theorem 32.6 in [1], if $(\mathbb{Z}_2)^k$ acts smoothly on $RP(2)$ with isolated fixed points, then there are at least two isolated fixed points. Since $\chi(RP(2)) = \chi(F) = 1$, the number of isolated fixed points is odd and at least three.

(1) If there are three of fixed points, y_1, y_2, and y_3 whose local representation classes are distinct, then by decomposition of local representation class, we know that the fixed point data at y_1, y_2 and y_3 are distinct.

(2) If there are only two local representation classes, then for $X(y_1) = X(y_2)$, we can cancel off these two fixed points without changing equivariant class $[(\mathbb{Z}_2)^k, RP(2)] \in \mathbb{Z}_2((\mathbb{Z}_2)^k)$ (see [1]), and so we use this method to cancel off these points whose local representation classes are the same. Finally we obtain $((\mathbb{Z}_2)^k, V^2)$ with only one isolated fixed point and the equivariant class $[(\mathbb{Z}_2)^k, V^2] = [(\mathbb{Z}_2)^k, RP(2)]$. This contradicts to Theorem 32.6 in [1].

(3) If there are only one local representation classes, then similarly we also get a contradiction.

Hence $\| J^2_{*,k} \| \geqslant 3$, $k \geqslant 2$, and the result follows. $\square$

4. $\| J^3_{*,k} \|$ ($k \geqslant 2$)

Before proving Theorem 1.2, let us introduce some lemmas.

Lemma 4.1. *([8]) For a $(\mathbb{Z}_2)^2$-action (M^n, T), the manifold M^n is bordant to the union of $RP((\lambda \otimes \varepsilon_3) \oplus \varepsilon_1 \oplus R)$ and $RP((\lambda \otimes \varepsilon_2) \oplus \lambda \oplus \varepsilon_1 \oplus R)$, where the first fibers over $RP(\varepsilon_2 \oplus R)$ and then over F, and the second fibers over $RP(\varepsilon_3)$ and then over F.*

Lemma 4.2. *([1]) Let $\xi^k \to V^n$ be a smooth k-plane bundle over a closed manifold. If either $k = 2$ or $n = 1$, then $[RP(\xi^k)] = 0$.*

Let $I = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, a, b, c, \nu\}$, where $\tau_1 = (2,1,0)$, $\tau_2 = (1,0,2)$, $\tau_3 = (0,2,1)$, $\tau_4 = (1,2,0)$, $\tau_5 = (0,1,2)$, $\tau_6 = (2,0,1)$, $a = (3,0,0)$, $b = (0,3,0)$, $c = (0,0,3)$ and $\nu = (1,1,1)$.

In the following we take $x_4 = [RP(1,1,0)]$, then the Stiefel–Whitney number $w_1^4(x_4) = 1$. By Lemma 2.3, $x_4 \in J^3_{*,2}$.

Lemma 4.3. *If $A \subseteq I$ and $\|A\| = 1$, then $x_4 \notin J^3_{*,2}(A)$ and $J^3_{*,2}(A) \neq J^3_{*,2}$.*

Proof. (1) If $A = \{\tau_i\}$, $i = 1,2,\dots,6$, by Theorem 4.2 in [2], $J^3_{*,2}(A) = 0$.

(2) If $A = \{a\}, \{b\}$ or $\{c\}$, by [2] and [9], $J^3_{*,2}(A) = J^3_{*,1} = \{\alpha \in MO_n \mid w_1^j w_{n-j}(\alpha) = w_1^{i-5} w_{n-i} s_5(\alpha) = 0,\ 5 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant n,\ n \geqslant 3\}$.

(3) If $A = \{\nu\}$, by Theorem 4.1 in [2],

$$J^3_{*,2}(A) = \big\{\alpha \in MO_n \bigm| w_1^i w_{n-i}(\alpha) = 0,\ i = 0,1,2,\dots,n,\ n \geqslant 3\big\}.$$

From $w_1^4(x_4) = 1$, we know that if $\|A\| = 1$, $x_4 \notin J^3_{*,2}(A)$. Hence $J^3_{*,2}(A) \neq J^3_{*,2}$. □

Lemma 4.4. *If $A \subseteq \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6\}$ and $\|A\| = 2$, then $x_4 \notin J^3_{*,2}(A)$ and $J^3_{*,2}(A) \neq J^3_{*,2}$.*

Proof. According to Theorem 4.1 in [2], if $A = \{\tau_1, \tau_2\}, \{\tau_1, \tau_3\}, \{\tau_1, \tau_4\}, \{\tau_1, \tau_6\}, \{\tau_2, \tau_3\}, \{\tau_2, \tau_5\}, \{\tau_2, \tau_6\}, \{\tau_3, \tau_4\}, \{\tau_3, \tau_5\}, \{\tau_4, \tau_5\}, \{\tau_4, \tau_6\}$ or $\{\tau_5, \tau_6\}$, then $J^3_{*,2}(A) = 0$.

Let $A = \{\tau_3, \tau_6\}$. If $x_4 \in J^3_{*,2}(A)$, by Lemma 4.1 x_4 is represented by the union of the following fiberings:

(1) $RP(\lambda \otimes \varepsilon_3 \oplus R) \to RP(\varepsilon_2 \oplus R) \to F_{\tau_3}$;
(2) $RP(\lambda \otimes \varepsilon_2 \oplus \lambda \oplus R) \to RP(\varepsilon_3) \to F_{\tau_3}$;
(3) $RP(\lambda \otimes \varepsilon_3 \oplus \varepsilon_1 \oplus R) \to F_{\tau_6}$;
(4) $RP(\lambda \oplus \varepsilon_1 \oplus R) \to RP(\varepsilon_3) \to F_{\tau_6}$.

From $\dim(\lambda \otimes \varepsilon_3 \oplus R) = 2$ and $\dim(RP(\varepsilon_3)) = \dim(F_{\tau_6}) = 1$, by Lemma 4.2 we know that the classes of (1)–(4) are zero, i.e. $[RP(\lambda \otimes \varepsilon_3 \oplus R)] = [RP(\lambda \otimes \varepsilon_2 \oplus \lambda \oplus R)] = [RP(\lambda \otimes \varepsilon_3 \oplus \varepsilon_1 \oplus R)] = [RP(\lambda \oplus \varepsilon_1 \oplus R)] = 0$. On the other hand, $x_4 \neq 0$, this is a contradiction. So $x_4 \notin J^3_{*,2}(A)$ for $A = \{\tau_3, \tau_6\}$. Let $\sigma_1 = (132)$ and $\sigma_2 = (123)$. Then $\sigma_1 A = \{\tau_2, \tau_4\}$, $\sigma_2 A = \{\tau_1, \tau_5\}$. By Lemma 2.4, $x_4 \notin J^3_{*,2}(\sigma_1 A) = J^3_{*,2}(\sigma_2 A) = J^3_{*,2}(A)$.

Hence if $A \subseteq \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6\}$ and $\|A\| = 2$, then $x_4 \notin J^3_{*,2}(A)$ and $J^3_{*,2}(A) \neq J^3_{*,2}$. □

Lemma 4.5. *If $A = \{a, \tau_i\}, \{b, \tau_i\}, \{c, \tau_i\}$, $i = 1,2,\dots,6$, then $x_4 \notin J^3_{*,2}(A)$ and $J^3_{*,2}(A) \neq J^3_{*,2}$.*

Proof. Let $A = \{a, \tau_1\}$. If $x_4 \in J^3_{*,2}(A)$, by Lemma 4.1 x_4 is represented by the union of the following fiberings:

(1) $RP(\varepsilon_1 \oplus R) \to F_a$;
(2) $RP(\lambda \oplus \varepsilon_1 \oplus R) \to F_a$;
(3) $RP(\varepsilon_1 \oplus R) \to RP(\varepsilon_2 \oplus R) \to F_{\tau_1}$.

According to Lemma 4.2, the classes of (1) and (2) are zero. For (3), let Stiefel–Whitney class $w(\varepsilon_1 \oplus R) = w(\varepsilon_1) = 1 + A_1 + A_2$, $w_1(\varepsilon_2 \oplus R) = w_1(\varepsilon_2) = 1 + v$, and c (respectively d) the characteristic class of λ (respectively of the line bundle over $RP(\varepsilon_1 \oplus R)$). Then

$$\begin{aligned}
w\big(RP(\varepsilon_1 \oplus R)\big) &= w(F_{\tau_1})\big[(1+c)^2 + (1+c)v\big]\big[(1+d)^3 + (1+d)^2 A_1 + (1+d)A_2\big] \\
&= \big(1 + c^2 + cv + v\big)\big(1 + d + d^2 + d^3 + A_1 + d^2 A_1 + A_2 + dA_2\big) \\
&= (1+v)\big(1 + d + d^2 + A_1 + A_2\big) \\
&= 1 + d + A_1 + v + d^2 + A_2 + dv + A_1 v + d^2 v + A_2 v.
\end{aligned}$$

Because ε_1 is the pullback from F, $A_1^2 = 0$ and $A_2 = 0$.

$$w_1^4\big(RP(\varepsilon_1 \oplus R)\big) = (d + A_1 + v)^4 = d^4 = d \cdot d^3 = d\big(d^2 A_1\big) = d^3 A_1 = d^2 A_1^2 = 0.$$

So $w_1^4(x_4) = 0$, this is a contradiction. Thus $x_4 \notin J^3_{*,2}(A)$.

Let $A = \{a, \tau_2\}$. If $x_4 \in J^3_{*,2}(A)$, then by Lemma 4.1 x_4 is represented by the union of the following fiberings:

(1) $RP(\varepsilon_1 \oplus R) \to F_a$;
(2) $RP(\lambda \oplus \varepsilon_1 \oplus R) \to F_a$;

(3) $RP(\lambda \otimes \varepsilon_3 \oplus \varepsilon_1 \oplus R) \to F_{\tau_2}$;
(4) $RP(\lambda \oplus \varepsilon_1 \oplus R) \to RP(\varepsilon_3) \to F_{\tau_2}$.

By Lemma 4.2, the classes of (1)–(3) are zero. The total Stiefel–Whitney class of (4) is

$$\begin{aligned} w\big(RP(\lambda \oplus \varepsilon_1 \oplus R)\big) &= w(F_{\tau_2})\big[(1+c)^2 + (1+c)v\big]\big[(1+d)^3 + (1+d)^2A_1 + (1+d)A_2\big] \\ &= \big(1 + c^2 + cv + v\big)\big(1 + d + d^2 + d^3 + A_1 + d^2A_1 + A_2 + dA_2\big) \\ &= (1+v)\big(1 + d + d^2 + A_1 + A_2\big) \\ &= 1 + d + A_1 + v + d^2 + A_2 + dv + A_1v + d^2v + A_2v, \end{aligned}$$

where c (respectively d) is the characteristic class of λ (respectively of the line bundle over $RP(\lambda \oplus \varepsilon_1 \oplus R)$), $v = w_1(\varepsilon_3)$, and $A_i = w_i(\lambda \oplus \varepsilon_1 \oplus R)$.

Let $w_1(\varepsilon_1) = u$. Then $A_1 = c + u$ and $A_2 = cu$, $w_1^4(RP(\lambda \oplus \varepsilon_1 \oplus R)) = (d + A_1 + v)^4 = d^4 = d \cdot d^3 = d(d^2A_1 + dA_2) = d^3A_1 + d^2A_2 = d^2(A_1^2 + A_2)$. By [10] and [14], $w_1^4[RP(\lambda \oplus \varepsilon_1 \oplus R)] = d^2(A_1^2 + A_2)[RP(\lambda \oplus \varepsilon_1 \oplus R)] = (A_1^2 + A_2)[RP(\varepsilon_3)] = ((c+u)^2 + cu)[RP(\varepsilon_3)] = (c^2 + cu)[RP(\varepsilon_3)] = c(v+u)[RP(\varepsilon_3)] = (v+u)[F_{\tau_2}] = 0$. So $w_1^4(x_4) = 0$, this is a contradiction. Thus $x_4 \notin J^3_{*,2}(A)$.

For $A = \{a, \tau_3\}, \{a, \tau_4\}, \{a, \tau_5\}$ and $\{a, \tau_6\}$, similarly we can obtain $x_4 \notin J^3_{*,2}(A)$.

Hence if $A = \{a, \tau_i\}$, $i = 1, 2, \dots, 6$, then $x_4 \notin J^3_{*,2}(A)$ and $J^3_{*,2}(A) \neq J^3_{*,2}$. According to Lemma 2.4, if either $A = \{b, \tau_i\}$ or $\{c, \tau_i\}$, $i = 1, 2, \dots, 6$, then $x_4 \notin J^3_{*,2}(A)$ and $J^3_{*,2}(A) \neq J^3_{*,2}$. □

Lemma 4.6. *If $A = \{\nu, \tau_i\}, \{\nu, a\}, \{\nu, b\}$ or $\{\nu, c\}$, $i = 1, 2, \dots, 6$, then $x_4 \notin J^3_{*,2}(A)$ and $J^3_{*,2}(A) \neq J^3_{*,2}$.*

Proof. Let $A = \{\nu, \tau_6\}$. If $x_4 \in J^3_{*,2}(A)$, by Lemma 4.1 x_4 is represented by the union of the following fiberings:

(1) $RP(\lambda \otimes \varepsilon_3 \oplus \varepsilon_1 \oplus R) \to RP(\varepsilon_2 \oplus R) \to F_\nu$;
(2) $RP(\lambda \otimes \varepsilon_2 \oplus \lambda \oplus \varepsilon_1 \oplus R) \to RP(\varepsilon_3) \to F_\nu$;
(3) $RP(\lambda \otimes \varepsilon_3 \oplus \varepsilon_1 \oplus R) \to F_{\tau_6}$;
(4) $RP(\lambda \oplus \varepsilon_1 \oplus R) \to RP(\varepsilon_3) \to F_{\tau_6}$.

By Lemma 4.2, the classes of (2)–(4) are zero. For (1), similar arguments show that $w_1^4[RP(\lambda \otimes \varepsilon_3 \oplus \varepsilon_1 \oplus R)] = 0$. So $w_1^4(x_4) = 0$, this is a contradiction. Thus $x_4 \notin J^3_{*,2}(A)$. By Lemma 2.4, if $A = \{\nu, \tau_i\}$, $i = 1, 2, \dots, 6$, then $x_4 \notin J^3_{*,2}(A)$. Similarly, if $A = \{a, \nu\}, \{b, \nu\}$ or $\{c, \nu\}$, then $x_4 \notin J^3_{*,2}(A)$ and $J^3_{*,2}(A) \neq J^3_{*,2}$. □

Lemma 4.7. *If $A \subseteq \{(3,0,0), (0,3,0), (0,0,3)\}$ and $\|A\| = 2$, then $x_4 \notin J^3_{*,2}(A)$ and $J^3_{*,2}(A) \neq J^3_{*,2}$.*

Proof. If $A \subseteq \{(3,0,0), (0,3,0), (0,0,3)\}$, from [2] we know that $J^3_{*,2}(A) = J^3_{*,1} = \{\alpha \in MO_n \mid w_1^j w_{n-j}(\alpha) = w_1^{i-5} w_{n-i} s_5(\alpha) = 0,\ 5 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant n,\ n \geqslant 3\}$. So $x_4 \notin J^3_{*,2}(A)$ and $J^3_{*,2}(A) \neq J^3_{*,2}$. □

Proof of Theorem 1.2. (1) Let $A = \{(2,1,0), (2,0,1), (1,1,1)\}$. By Theorem 2 in [11], $J^3_{*,2}(A) = J^3_{*,2}$, hence $\|J^3_{*,2}\| \leqslant 3$. According to Lemmas 4.3–4.7, $\|J^3_{*,2}\| \geqslant 3$, hence $\|J^3_{*,2}\| = 3$.

(2) By Lemma 2.5 and (1), $\|J^3_{*,k}\| \leqslant \|J^3_{*,2}\| = 3$ $(k \geqslant 3)$. □

5. $\|J^4_{*,k}\|$ $(k \geqslant 2)$ and $\|J^5_{*,2}\|$

Let $B = \{(3,1,1), (3,2,0), (3,0,2), (2,1,2), (2,2,1), (1,2,2), (4,0,1), (4,1,0), (5,0,0)\}$. Then we have

Lemma 5.1. $[RP(5, n_2, n_3)] \in J^5_{*,2}(B)$.

Proof. Let $T_{1,1}$ be an involution on $RP(5)$ with the fixed point set $RP(2) \cup RP(2)$ and $\widetilde{T_{1,1}}$ an involution on λ_1 covering $T_{1,1}$. $RP(n_2)$ and $RP(n_3)$ admit the trivial action of $(\mathbb{Z}_2)^0$. Define T_1, T_2 to be the involution on $RP(5, n_2, n_3)$ induced by $-1 \times 1 \times 1$ and $\widetilde{T_{1,1}} \times -1 \times 1$ on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ respectively. Then (T_1, T_2) defines a $(\mathbb{Z}_2)^2$-action on $RP(5, n_2, n_3)$ with the fixed point set $F = \{RP(2) \cup RP(2)\} \times RP(n_2) \times RP(n_3) \times \{e_1, e_2, e_3\}$. The components of F correspond to $\{(3,1,1), (4,1,0), (4,0,1)\}$, hence $[RP(5, n_2, n_3)] \in J^5_{*,2}(B)$. □

Lemma 5.2. $[RP(n_1, n_2, \dots, n_{10})] \in J^5_{*,2}(B)$.

Proof. Let

$$T_1[y_1, y_2, \ldots, y_5, y_6, \ldots, y_{10}] = [-y_1, \quad -y_2, \quad \ldots, \quad -y_5, \quad y_6, \quad \ldots, \quad y_{10}],$$
$$T_2[y_1, y_2, \ldots, y_5, y_6, \ldots, y_{10}] = [y_1, \quad y_2, \quad \ldots, \quad y_5, \quad y_6, \quad \ldots, \quad y_{10}].$$

Then (T_1, T_2) defines a $(\mathbb{Z}_2)^2$-action on $RP(n_1, n_2, \ldots, n_{10})$ with the fixed point set

$$F = RP(n_1, n_2, \ldots, n_5) \cup RP(n_6, n_7, \ldots, n_{10}),$$

and

$$F_{T_1} = RP(n_1, n_2, \ldots, n_5) \cup RP(n_6, n_7, \ldots, n_{10}),$$
$$F_{T_2} = RP(n_1, n_2, \ldots, n_{10}),$$
$$F_{T_1T_2} = RP(n_1, n_2, \ldots, n_5) \cup RP(n_6, n_7, \ldots, n_{10}).$$

The components of F correspond to $\{(5, 0, 0)\}$, therefore $[RP(n_1, n_2, \ldots, n_{10})] \in J^5_{*,2}(B)$. □

Lemma 5.3. *There exist indecomposable classes $x_n \in J^5_{*,2}(B)$ for $n \geqslant 8$.*

Proof. By [13] and Lemmas 5.1–5.2, the lemma is established. □

Proof of Theorem 1.3. *Case* 1. $r = 4$ and $k = 2$.

Let $C = \{(2, 2, 0), (2, 0, 2), (0, 2, 2), (1, 2, 1), (2, 1, 1), (1, 1, 2), (1, 3, 0), (0, 3, 1)\}$. By exhibiting special generators of MO_*, we show $J^4_{*,2}(C) = J^4_{*,2}$, hence $\| J^4_{*,2} \| \leqslant 8$.

Take a system of generators of MO_* as follows:

(i) Let $x_2 = [RP(2)]$, $x_4 = [RP(1, 1, 0)]$. Then $x_2, x_4 \in J^2_{*,2}(A)$, where $A = \{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$.

(ii) Take $x_5 = [RP(2, 0, 0, 0)]$ and $x_6 = [RP(3, 0, 0, 0)]$. According to Lemma 2.7 and Lemma 2.8 in [12], x_5 and $x_6 \in J^4_{*,2}(C)$.

(iii) For $n = 2^u > 4$, take $x_n = [RP(3, 2^u - 5, 0)]$. In order to show $x_n \in J^4_{*,2}(C)$, we construct a $(\mathbb{Z}_2)^2$-action on $RP(3, 2^u - 5, 0)$. Let T be an involution on $RP(3)$ with two copies of $RP(1)$ as the fixed point set and $\widetilde{T}$ an involution on λ_1 covering T. For $RP(3, 2^u - 5, 0)$, let

$$T_1[y_1, y_2, y_3] = [-\widetilde{T} y_1, \quad y_2, \quad y_3],$$
$$T_2[y_1, y_2, y_3] = [\widetilde{T} y_1, \quad -y_2, \quad y_3].$$

Then (T_1, T_2) defines a $(\mathbb{Z}_2)^2$-action on $RP(3, 2^u - 5, 0)$ with the fixed point set

$$F = \{RP(1) \cup RP(1)\} \times RP\big(2^u - 5\big) \times RP(0) \times \{e_1, e_2, e_3\}.$$

The components of F correspond to $\{(1, 2, 1)(1, 3, 0)(0, 3, 1)\}$. So $x_n \in J^4_{*,2}(C)$.

(iv) For $n > 8, n \neq 2^u - 1$ and 2^u, by Lemma 2.4 in [12], there are indecomposable classes $x_n \in J^4_{*,2}(C)$.

From Lemma 2.4, we know that x_2^2, x_4^2 and $x_2x_4 \in J^4_{*,2}(C)$. By [7], $x_4 \notin J^4_{*,2}$, so $x_4 \notin J^4_{*,2}(C)$. Since $J^4_{*,2}(C)$ is an ideal in MO_*, $J^4_{*,2}(C)$ consists of 4-dimensional decomposable classes and all n-dimensional classes with $n > 4$. By [7], $J^4_{*,2}(C) = J^4_{*,2}$.

Case 2. $r = 4$ and $k = 3$.

Let $D = \{(1, 1, 1, 0, 1, 0, 0), (0, 1, 0, 1, 0, 1, 1), (1, 0, 1, 0, 1, 0, 1), (1, 0, 1, 1, 1, 0, 0), (1, 0, 1, 0, 1, 1, 0), (2, 2, 0, 0, 0, 0, 0), (0, 2, 2, 0, 0, 0, 0), (2, 0, 2, 0, 0, 0, 0)\}$. By exhibiting special generators of MO_*, we show $J^4_{*,3}(D) = \sum_{n \geqslant 4}^{\infty} MO_n = J^4_{*,3}$, hence $\| J^4_{*,3} \| \leqslant 8$.

Let $x_2 = [RP(2)]$, $x_5 = [RP(2, 0, 0, 0)]$. By Lemma 2.5 and Lemma 2.7 in [12], x_2^2 and $x_5 \in J^4_{*,2}(P) \subseteq J^4_{*,3}$, where $P = \{(2, 2, 0), (2, 0, 2), (0, 2, 2)\}$. According to Lemma 2.4, x_2^2 and $x_5 \in J^4_{*,3}(P \oplus \theta) \subseteq J^4_{*,3}(D)$, where $P \oplus \theta = \{(2, 2, 0, 0, 0, 0, 0), (2, 0, 2, 0, 0, 0, 0), (0, 2, 2, 0, 0, 0, 0)\}$. For $n \geqslant 4$, $n \neq 5$, and $n \neq 2^u - 1$, by Corollary 3.2 in [6], there exist indecomposable classes $x_n = [RP(n_1, n_2, n_3, n_4, n_5)]$. For $RP(n_1, n_2, n_3, n_4, n_5)$, let

$$T_1[y_1, y_2, y_3, y_4, y_5] = [y_1, \quad y_2, \quad -y_3, \quad y_4, \quad -y_5],$$
$$T_2[y_1, y_2, y_3, y_4, y_5] = [y_1, \quad -y_2, \quad y_3, \quad y_4, \quad y_5],$$
$$T_3[y_1, y_2, y_3, y_4, y_5] = [y_1, \quad y_2, \quad -y_3, \quad -y_4, \quad y_5].$$

Then (T_1, T_2, T_3) defines a $(\mathbb{Z}_2)^3$-action on $RP(n_1, n_2, \dots, n_5)$ with the fixed point set

$$F = RP(n_1) \times RP(n_2) \times RP(n_3) \times RP(n_4) \times RP(n_5) \times \{e_1, e_2, e_3, e_4, e_5\},$$

and

$$\begin{aligned}
F_{T_1,T_2} &= RP(\lambda_1 \oplus \lambda_4) \cup RP(\lambda_3 \oplus \lambda_5) \cup RP(\lambda_2),\\
F_{T_1,T_3} &= RP(\lambda_1 \oplus \lambda_2) \cup RP(\lambda_3) \cup RP(\lambda_4) \cup RP(\lambda_5),\\
F_{T_2,T_3} &= RP(\lambda_1 \oplus \lambda_5) \cup RP(\lambda_3 \oplus \lambda_4) \cup RP(\lambda_2),\\
F_{T_1,T_2T_3} &= RP(\lambda_1) \cup RP(\lambda_2 \oplus \lambda_4) \cup RP(\lambda_3) \cup RP(\lambda_5),\\
F_{T_2,T_1T_3} &= RP(\lambda_1 \oplus \lambda_3) \cup RP(\lambda_4 \oplus \lambda_5) \cup RP(\lambda_2),\\
F_{T_3,T_1T_2} &= RP(\lambda_1) \cup RP(\lambda_2 \oplus \lambda_5) \cup RP(\lambda_3) \cup RP(\lambda_4),\\
F_{T_1T_2,T_2T_3} &= RP(\lambda_1) \cup RP(\lambda_2 \oplus \lambda_3) \cup RP(\lambda_4) \cup RP(\lambda_5).
\end{aligned}$$

The components of F correspond to $\{(1,1,1,0,1,0,0), (0,1,0,1,0,1,1), (1,0,1,0,1,0,1), (1,0,1,1,1,0,0), (1,0,1,0,1,1,0)\}$. Thus $x_n \in J^4_{*,3}(D)$ for $n \geqslant 4$, $n \neq 5$, and $n \neq 2^u - 1$.

Since $J^4_{*,3}(D)$ is an ideal in MO_*, $J^4_{*,3}(D) = \sum_{n \geqslant 4}^{\infty} MO_n = J^4_{*,3}$.

Case 3. $r = 4$ and $k \geqslant 4$.

By Lemma 2.5, $\|J^4_{*,k}\| \leqslant \|J^4_{*,3}\| \leqslant 8$ $(k \geqslant 4)$.

Case 4. $r = 5$ and $k = 2$.

Take $x_2 = [RP(2)]$, $x_4 = [RP(1,1,0)]$, $x_5 = [RP(2,0,0,0)]$ and $x_6 = [RP(3,0,0,0)]$. For $n \geqslant 8$, take x_n as in Lemma 5.3. By [6], $x_2^i \notin J^5_{*,2}$, so $x_2^i \notin J^5_{*,2}(B)$, where $B = \{(3,1,1), (3,2,0), (3,0,2), (2,1,2), (2,2,1), (1,2,2), (4,0,1), (4,1,0), (5,0,0)\}$.

Since $x_2, x_4, x_5, x_6 \in J^2_{*,2}(P)$, where $P = \{(1,0,1), (1,1,0), (0,1,1)\}$, and $x_4, x_5, x_6 \in J^3_{*,2}(Q)$, where $Q = \{(2,0,1), (2,1,0), (1,1,1)\}$, by Lemma 2.4, $x_2x_4, x_2x_5, x_2x_6, x_4x_5, x_4x_6, x_5x_6, x_4^2, x_5^2, x_6^2 \in J^5_{*,2}(B)$. By [13],

$$J^5_{n,2}(B) = J^5_{n,2} = \begin{cases} MO_n \cap \ker\chi, & n \geqslant 7,\\ D_n \cap \ker\chi, & n = 5, 6, \end{cases}$$

where $\chi : MO_* \to \mathbb{Z}_2$ denotes the mod 2 Euler characteristic. Hence $J^5_{*,2}(B) = J^5_{*,2}$, $\|J^5_{*,2}\| \leqslant 9$. $\square$

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